Journal of Sound and Vibration (2000) **233**(5), 877–896
doi:10.1006/jsvi.1999.2843, available online at http://www.idealibrary.com on IDEAL®

RECONSTRUCTION OF LINEAR AND NON-LINEAR CONTINUOUS-TIME SYSTEM MODELS FROM INPUT/OUTPUT DATA USING THE KERNEL INVARIANCE ALGORITHM

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(*Received* 29 *March* 1999, *and in final form* 10 *December* 1999)

A new kernel invariance algorithm (KIA) is introduced to determine both the significant model terms and estimate the unknown parameters in non-linear continuous-time differential equation models of unknown systems

1. INTRODUCTION

Although most physical systems are continuous in nature the input–output data from these systems is usually sampled and a discrete-time model is identified. But in some cases a continuous-time model, which is often easier to relate to the physical operation of the underlying system, is required. Identification of linear continuous-time models has been studied by several authors and can be classified into direct and indirect approaches [1]. The direct approach is based on the output error or equation error methods [2]. The equation error method has been widely employed and is based on converting the differential equations into a linear algebraic form. Modulating functions, orthogonal polynomials and linear integral filters have been used in the literature [3–7]. The indirect approach consists of fitting either a non-parametric model (step, impulse or frequency response) or a discrete parametric ARMAX model initially and then constructing a continuous-time model of the system from this [2, 8, 9].

In the linear time-invariant case, the "impulse invariance method" (IIM) [10] is based upon the equivalence between the linear time-invariant differential and difference equations. Zhao and Marmarelis [11] recently extended this basic concept to non-linear time-invariant models and called the new approach the "kernel invariance method" (KLM). The method exploits the equivalence between the high order kernels associated with non-linear differential and difference equation models. The great advantage is that this approach avoids the direct computation of derivatives which can induce severe numerical problems and the non-linear model can be constructed sequentially by building in the linear model terms, followed by the quadratic terms and so on.

Identification of continuous-time non-linear differential equation models from sampled data is an important problem that has only been studied by a few authors [12, 13]. The KIM offers one possible solution to this problem and in the present study the method is developed into a practical identification procedure. In the original formulation by Zhao and

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Marmarelis all the calculations were done by hand, the authors noted that the analysis "can be a rather unwieldly task in general as demonstrated by two relatively simple examples", and no account was taken of noise effects and bias. But in the identification of practical non-linear systems almost all these restrictions will be violated because the discrete-time non-linear model that is identified from the input–output data is often complex and can involve many terms. A new practical procedure is therefore introduced below which uses a new orthorgonalised version of the generalised least-squares algorithm [14] to select the significant model terms and to yield unbiased estimates of the parameters in continuous-time non-linear differential equation models. The new method will be refered to as the kernel invariance algorithm (KIA).

The paper is organised as follows. In section 2, the basic concepts of non-linear system representations and the KIM are introduced. In section 3, the reconstruction formulation for linear and non-linear continuous-time models from difference equation models is described, and an orthogonal least-squares procedure is introduced to determine the model structure. In section 4, a simulation example is used to illustrate the identification procedure, and in section 5 a real application of an electromagnet bearing control system is described. Finally, conclusions are given in section 6.

2. THE KERNEL INVARIANCE METHOD (KIM)

A wide class of continuous-time non-linear systems can be represented by the Volterra functional series [15],

$$y(t) = \sum_{n=1}^{N} y_n(t), \tag{1}$$

where $y_n(t)$ is the nth order output of the system

$$y_n(t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, \ldots, \tau_n) \prod_{i=1}^{n} u(t-\tau_i)\, \mathrm{d}\tau_i, \quad n > 0 \tag{2}$$

$u(t)$ is the input and $h_n(\tau_1, \ldots, \tau_n)$ is called the nth order Volterra kernel or impulse response function. If $n = 1$, this reduces to the familiar linear impulse response function.

In the KIM introduced by Zhao and Marmarelis [11] non-linear systems described by non-linearities of only second degree were considered. This was presumably because of the complexity associated with higher order non-linear effects. However, results are available in the literature which can be applied immediately to extend these ideas to the much more realistic and general non-linear case. These results form the basis of the new KIA and are reviewed below.

Many continuous-time systems can also be characterised by a non-linear differential equation (NDE) model

$$f\left(y, \frac{\mathrm{d}y}{\mathrm{d}t}, \ldots, \frac{\mathrm{d}^p y}{\mathrm{d}t^p};\; u, \frac{\mathrm{d}u}{\mathrm{d}t}, \ldots, \frac{\mathrm{d}^q u}{\mathrm{d}t^q}\right) = 0. \tag{3}$$

The polynomial form of equation (3) is given by the model

$$\sum_{n=1}^{N} \sum_{p=0}^{n} \sum_{l_1, l_{p+q}=0}^{L} c_{p,q}(l_1, \ldots, l_{p+q}) \prod_{i=1}^{p} D^{l_i} y(t) \prod_{i=p+1}^{p+q} D^{l_i} u(t) = 0, \tag{4}$$

where q and p are the number of input and output terms, respectively, with $p + q = n$, L is the highest derivative of the input–output and $c_{p,q}(\cdot)$ represent the model parameters. The operator D is defined by

$$D^l x(t) = \frac{\mathrm{d}^l x(t)}{\mathrm{d}t^l}, \quad l \geqslant 0.$$

The nth order Volterra kernel can be related to the parameters of the NDE model. In fact, the multidimensional Laplace transform of the nth order kernel can be shown to be a function of the NDE model parameters [16]:

$$\begin{aligned} H_n^{asym}&(s_1, \dots, s_n) \\ &= \frac{1}{-\left[\sum_{l_1=0}^{L} c_{1,0}(l_1)(s_1 + \cdots + s_n)^{l_1}\right]} \left\{ \sum_{l_1, J_n = 1}^{L} c_{0,n}(l_1, \dots, l_n)\, s_1^{l_1} \cdots s_n^{l_n} \right. \\ &\quad + \sum_{q=1}^{n-1} \sum_{p=1}^{n-q} \sum_{l_1, J_n = 1}^{L} c_{p,q}(l_1, \dots, l_{p+q}) \times s_{n-q+1}^{l_{n-q+1}} \cdots s_{p+q}^{l_{p+q}}\, H_{n-p,q}^{asym}(s_1, \dots, s_{n-q}) \\ &\quad \left. + \sum_{p=2}^{n} \sum_{l_1, l_p = 0}^{L} c_{p,0}(l_1, \dots, l_p)\, H_{n,p}^{asym}(s_1, \dots, s_n) \right\}, \end{aligned} \tag{5}$$

where

$$H_{n,p}^{asym}(s_1, \dots, s_n) = \sum_{i=1}^{n-p+1} H_i(s_1, \dots, s_i)\, H_{n-i,p-1}^{asym}(s_{i+1}, \dots, s_n)\,(s_1 + \cdots + s_i)^{l_p} \tag{6}$$

and, without loss of generality, we assume $c_{1,0}(0) = -1$.

A commonly used non-linear discrete-time system model is the NARX model

$$y(k) = F[y(k-1), \dots, y(k-d_y), u(k-1), \dots, u(k-d_u)], \tag{7}$$

where $F[\cdot]$ represents some non-linear function of the lagged inputs $u(k-1), \dots, u(k-d_u)$ and outputs $y(k-1), \dots, y(k-d_y)$. Selecting $F[\cdot]$ to be a polynomial expression yields

$$y(k) = \sum_{m=1}^{M} y_m(k), \tag{8}$$

where M is the order of the non-linearity and $y_m(k)$, the mth order output of the system, is given by

$$y_m(k) = \sum_{p=0}^{m} \sum_{d_1, d_{p+q} = 1}^{K} b_{p,q}(d_1, \dots, d_{p+q}) \prod_{i=1}^{p} y(k-d_i) \prod_{i=p+1}^{p+q} u(k-d_i) \tag{9}$$

with

$$p + q = m,\ d_i = 1, \dots, K \quad \text{and} \quad \sum_{d_a, d_b = 1}^{K} \equiv \sum_{d_a = 1}^{K} \cdots \sum_{d_b = 1}^{K}, \tag{10}$$

where q and p are the number of input and output terms, respectively, and κ is the maximum lag of the input–output terms.

For the NARX model the multidimensional Z transform of the nth order kernel can be shown to be a function of the NARX model parameters [16]:

$$H_n^{asym}(z_1, \dots, z_n)$$
$$= \frac{1}{\left[1 - \sum_{k_1=1}^{K} b_{1,0}(k_1)(z_1 \cdots z_n)^{-k_1}\right]} \left\{ \sum_{k_1, k_n=1}^{K} b_{0,n}(k_1, \dots, k_n)\, z_1^{-k_1} \cdots z_n^{-k_n} \right.$$
$$+ \sum_{q=1}^{n-1} \sum_{p=1}^{n-q} \sum_{k_1, k_n=1}^{K} b_{p,q}(k_1, \dots, k_{p+q}) \times z_{n-q+1}^{-k_{n-q+1}} \cdots z_{p+q}^{-k_{p+q}}\, H_{n-p,q}^{asym}(z_1, \dots, z_{n-q})$$
$$\left. + \sum_{p=2}^{n} \sum_{k_1, k_p=1}^{K} b_{p,0}(k_1, \dots, k_p)\, H_{n,p}^{asym}(z_1, \dots, z_n) \right\}, \tag{11}$$

where

$$H_{n,p}^{asym}(z_1, \dots, z_n) = \sum_{i=1}^{n-p+1} H_i(z_1, \dots, z_i)\, H_{n-i,p-1}^{asym}(z_{i+1}, \dots, z_n)(z_1 \cdots z_i)^{-k_p}. \tag{12}$$

Equations (11) and (12) are the discrete-time equivalents to equations (5) and (6). The nth order transfer functions of equations (5) and (11) are not necessarily unique because changing the order of any two arguments generates a new function without changing the value of $y_n(t)$ in equations (1) and (8). However, the symmetric version of these functions are unique and these are given as

$$H_n^{sym}(s_1, \dots, s_n) = \frac{1}{n!} \sum_{\substack{all\ permutations \\ of\ \omega_1, \dots, \omega_n}} H_n^{asym}(s_1, \dots, s_n) \tag{13}$$

and

$$H_n^{sym}(z_1, \dots, z_n) = \frac{1}{n!} \sum_{\substack{all\ permutations \\ of\ \omega_1, \dots, \omega_n}} H_n^{asym}(z_1, \dots, z_n). \tag{14}$$

The KIM is based on the fact that the discrete high order kernels are the sampled versions of their continuous counterparts provided that the sampling interval is sufficiently short to avoid aliasing. This implies that

$$H_{n,p}^{sym}(s_1, \dots, s_n) = H_{n,p}^{sym}(z_1, \dots, z_n)_{z_1 = e^{s_1 T}, \dots, z_n = e^{s_n T}}, \tag{15}$$

where T is the sampling interval. If $n = 1$ in equation (15) this reduces to the well-known impulse invariance method (IIM) introduced by Oppenheim and Schafer [10]. Zhao and Marmarelis extended this to include both the linear model terms and the quadratic terms and called the new method the KIM. But the restriction to quadratic systems can be avoided and the results can be generalised to all analytic non-linear systems using the analysis introduced above. This follows because both the continuous-time and the discrete-time models have been related to the kernels in equation (15), so that the left-hand side of equation (15) $H_{n,p}^{sym}(s_1, \ldots, s_n)$ is equal to the expression on the right-hand side of equation (5) which is a function of the non-linear differential equation model coefficients $c_{p,q}(\cdot)$. Similarly, the right-hand side of equation (15) $H_{n,p}^{sym}(z_1, \ldots, z_n)_{z_1 = e^{s_1 T}, \ldots, z_n = e^{s_n T}}$ is equal to the right-hand side of equation (11) which is a function of the NARX model coefficients $b_{p,q}(\cdot)$. Therefore, if either set of coefficients is known, the other set can be determined. However, in system identification we are more likely to obtain the NARX model coefficients $b_{p,q}$ from sampled measurements of the input–output data. Once these coefficients have been estimated the equivalence in equation (15) can be used to construct the NDE model sequentially by building in the linear model terms followed by the quadratic terms and so on. In real applications the identified model is likely to be complex and the effects of noise or non-perfect estimates of the Kernel functions should be accommodated. Both these problems can be addressed by introducing the new KIA described below

3. RECONSTRUCTION FORMULATIONS

From equations (5) to (15), it is clear that the first order kernel is only related to the set of linear coefficients, the second order kernel is related to the linear and quadratic coefficients, the third order kernel is related to the linear, quadratic and cubic coefficients and so on. This suggests that the continuous-time model reconstruction procedure can be split and can be applied sequentially to reconstruct just the linear terms, followed by the quadratic terms, etc. An important problem at each construction stage is how to determine which of the many possible terms should be included in the continuous-time model. These issues will be investigated in the following section.

3.1. LINEAR CONTINUOUS TERMS RECONSTRUCTION

Consider initially the case $n = 1$ in equation (15) to yield the linear equivalence

$$H_1(z)_{|z = e^{sT}} = H_1(s) = \frac{\sum_{l_1=0}^{L} c_{0,1}(l_1)(s)^{l_1}}{1 + \sum_{l_1=1}^{L} c_{1,0}(l_1)(s)^{l_1}} = \frac{B(s)}{A(s)}. \tag{16}$$

The well-known map between the s- and the z-plane is illustrated in Table 1. Conventionally, the s-data is extracted along the imaginary (frequency) axis, that is, $s(1) = j\omega(1)$, $s(2) = j\omega(2), \ldots, s(N) = j\omega(N)$, where $H_1(s)$ is now called the frequency response, and this leads to algorithms solely in the frequency domain [9]. However, during the application of this approach to some real examples it was found that sometimes the results under this s-data selection criterion do not satisfy the mapping in the whole of the s-plane. In this paper therefore the s-data will be selected randomly along both the imaginary and the real axis of the s-plane to guarantee the mapping on both axes.

Assuming that a NARMAX model has been identified from sampled data records the linear transfer function can be computed from equation (11) to yield

$$\bar{H}_1(z) = H_1(z) + N_1(z), \tag{17}$$

where $\bar{H}_1(z)$ is obtained from the identified NARX model and $N_1(z)$ represents any inaccuracies or noise on $\bar{H}_1(z)$.

TABLE 1

Mapping the s-plane to the z-plane

s-plane	z-plane
$s = j\omega$ (frequency axis)	$\lvert z \rvert = 1$ unit circle
$s = \sigma \geqslant 0$	$z = r \geqslant 1$
$s = \sigma \leqslant 0$	$z = r$, $0 \leqslant r \leqslant 1$
$s = \sigma + jw$	$z = re^{j\theta}$ where $r = e^{\sigma T}$, $\theta = \omega T$

Thus,

$$\bar{H}_1(z)_{|z=e^{sT}} = \hat{H}_1(s) = \frac{\sum_{l_1=0}^{L} c_{0,1}(l_1)(s)^{l_1}}{1 + \sum_{l_1=1}^{L} c_{1,0}(l_1)(s)^{l_1}} + N_1(e^{sT})$$

$$= \frac{\sum_{l_1=0}^{L} c_{0,1}(l_1)(s)^{l_1}}{1 + \sum_{l_1=1}^{L} c_{1,0}(l_1)(s)^{l_1}} + N_1(s). \tag{18}$$

Equation (18) is a rational form with respect to s. As far as the estimation of $c_{0.1}(\cdot)$ and $c_{1,0}(\cdot)$ is concerned, s can be regarded as an input signal and $\bar{H}_1(z)_{|z=e^{sT}}$ or $\bar{H}_1(s)$ from the NARX model as a known output. So the problem is to estimate the unknown coefficients $c_{1,0}(\cdot)$, $c_{1,0}(\cdot)$ from a noisy rational process.

Assume that the noise process can be represented by the transfer function

$$N_1(s) = \frac{W(s)}{Q(s)}\,\xi_1(s) = \frac{\sum_{k_1=0}^{K_p} w(k_1)(s)^{k_1}}{\sum_{k_2=0}^{K_q} q(k_2)(s)^{k_2}}\,\xi_1(s), \tag{19}$$

where $\xi_1(s)$ is a zero mean white noise process. Multiplying out Eq. (18) and re-arranging gives

$$\hat{H}_1(s) = -\sum_{l_1=1}^{L} c_{1,0}(l_1)(s)^{l_1}\,\hat{H}_1(s) + \sum_{l_1=0}^{L} c_{0,1}(l_1)(s)^{l_1} + E_1(s), \tag{20}$$

where $E_1(s) = A(s)N_1(s) = (A(s)W(s)/Q(s))\,\xi_1(s)$.

Further arranging equation (20) yields

$$z(s) = \sum_{i=1}^{2L+1} \theta_i\,P(s) + E_1(s), \tag{21}$$

where

$$z(s) = \hat{H}_1(s),$$

$$\theta_1 = c_{0,1}(0), \qquad P_1 = 1,$$

$$\theta_2 = c_{0,1}(1), \qquad P_2 = s,$$

$$\vdots \qquad\qquad \vdots$$

$$
\begin{array}{ll}
\theta_{L+1} = c_{0,1}(L), & P_{L+1} = (s)^L, \\
\theta_{L+2} = c_{0,1}(1), & P_{L+2} = -(s)^1 \hat{H}_1(s), \\
\vdots & \vdots \\
\theta_{2L+1} = c_{1,0}(L), & P_{2L+1} = -(s)^L \hat{H}_1(s).
\end{array}
$$

If 'N' data points of $z(s)$ and $P_i(s)$ are available, at $s(i) = [R(i), \mathrm{j}I(i)]$, $i = 1, \ldots, N$, where $R(i)$ and $l(i)$ are random points, then equation (21) can be expressed as

$$
\mathbf{Z} = \mathbf{P}\boldsymbol{\Phi} + \boldsymbol{\Xi}, \tag{22}
$$

$$
\mathbf{Z} = \begin{bmatrix} z(s(1)) \\ z(s(2)) \\ \vdots \\ z(s(N)) \end{bmatrix}_{N \times 1}, \quad \boldsymbol{\Phi} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_{2L+1} \end{bmatrix}_{(2L+1) \times 1} \quad \boldsymbol{\Xi} = \begin{bmatrix} E_1(s(1)) \\ E_1(s(2)) \\ \vdots \\ E_1(s(N)) \end{bmatrix}_{N \times 1},
$$

where

$$
\mathbf{P} = \begin{bmatrix} P_1(s(1)) & P_2(s(1)) \cdots P_{2L+1}(s(1)) & \\ P_1(s(2)) & P_2(s(2)) \cdots P_{2L+1}(s(2)) & \\ \vdots & \vdots & \cdots \vdots \\ P_1(s(N)) & P_2(s(N)) \cdots P_{2L+1}(s(N)) & \end{bmatrix}_{N \times [2L+1)}.
$$

Finally, since $s(i) = [R(i), \mathrm{j}I(i)]$, $i = 1, \ldots, N$, are complex numbers, equation (22) should be partitioned into real and imaginary parts as

$$
\left[\frac{\mathrm{Re}(\mathbf{Z})}{\mathrm{lm}(\mathbf{Z})}\right] = \left[\frac{\mathrm{Re}(\mathbf{P})}{\mathrm{lm}(\mathbf{P})}\right] \boldsymbol{\Phi} + \left[\frac{\mathrm{Re}(\boldsymbol{\Xi})}{\mathrm{lm}(\boldsymbol{\Xi})}\right].
$$

This basic procedure will be applied to all the model reconstructions.

Although equation (22) is a linear-in-parameters expression, if least squares is applied directly the estimates would be biased because $E_1(s)$ is not white.

To overcome this problem, postulate a filter $F(s)$ and multiply both sides of equation (21) with $F(s)$ to give

$$
F(s)\,z(s) = \sum_{i=1}^{2L+1} \theta_i F(s) P(s) + F(s)\,[A(s) W(s) Q^{-1}(s)]\,\xi_1(s). \tag{23}
$$

If $F(s)$ is selected as

$$
F(s) = A(s)^{-1}\, W^{-1}(s)\, Q(s)
$$

then equation (23) becomes

$$
F(s)\,z(s) = \sum_{i=1}^{2L+1} \theta_i F(s) P(s) + \xi_1(s) \tag{24}
$$

or in matrix form

$$\mathbf{Z}_F = \mathbf{P}_F \boldsymbol{\Phi} + \varsigma, \tag{25}$$

$$\mathbf{Z}_F = \begin{bmatrix} F(s(1))z(s(1)) \\ F(s(2))z(s(2)) \\ \vdots \\ F(s(N))z(s(N)) \end{bmatrix}_{N\times 1}, \quad \boldsymbol{\Phi} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_{2L+1} \end{bmatrix}_{(2L+1)\times 1} \quad \varsigma = \begin{bmatrix} \xi_1(s(1)) \\ \xi_1(s(2)) \\ \vdots \\ \xi_1(s(N)) \end{bmatrix}_{N\times 1},$$

where

$$\mathbf{P}_F = \begin{bmatrix} F(s(1))\,P_1(s(1))\;F(s(1))\,P_2(s(1)) \cdots F(s(1))\,P_{2L+1}(s(1)) \\ F(s(2))\,P_1(s(2))\;F(s(2))P_2(s(2)) \cdots F(s(2))\,P_{2L+1}(s)2)) \\ \vdots \qquad\qquad \vdots \qquad\qquad \cdots\; \vdots \\ F(s(N))\,P_1(s(N))\;F(s(N))\,P_2(s(N)) \cdots F(s(N))\,P_{2L+1}(s(N)) \end{bmatrix}_{N\times(2L+1)}.$$

and the noise $E_1(s)$ is reduced to a white signal $\xi_1(s)$ and unbiased estimates of the system parameters will be obtained using least squares. This is essentially a modified version of the generalised least-squares (GLS) algorithm developed by Clarke [14].

The practical implementation of the above idea can be summarised in the following steps and is summarised in the flow chart in Figure 1.

(1) Randomly select $s(i) = [R(i), \mathrm{j}I(i)]$, $i = 1, \ldots, N$ in the s-plane and form $\mathbf{Z}$ and $\mathbf{P}$ in equation (22). Apply the standard linear least squares to obtain the initial estimates of $\hat{\boldsymbol{\Phi}}$ in equation (22). These estimates will be biased if the noise is coloured.

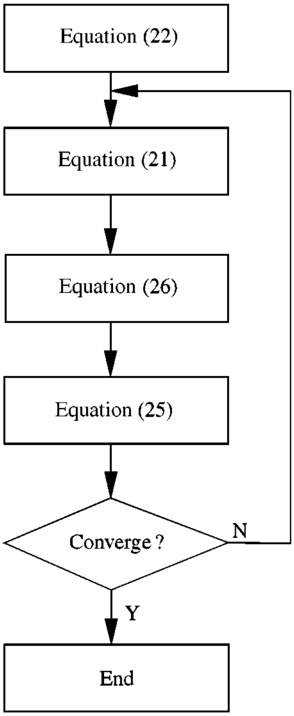

Figure 1. Flow chart of the reconstruction procedure for continuous-time linear term.

(2) Analyse the residual from equation (21):

$$\hat{E}_l(s) = \mathbf{z}(s) - \sum_{i=1}^{2L+1} \hat{\theta}_i \, P(s).$$

(3) Set the filter $F(s) = \sum_{k=0}^{G} f_k \, s^k$, $f_0 = 1$, where G is the order. Estimate the filter $\hat{F}(s)$ from $\hat{F}(s)\hat{E}_1(s) = \xi_1(s)$ using least squares. In matrix form this gives,

$$\mathbf{\Gamma} = \mathbf{\Psi\Theta} + \mathbf{\Delta}_1, \tag{26}$$

$$\mathbf{\Gamma} = \begin{bmatrix} \hat{E}(s(1)) \\ \hat{E}(s(2)) \\ \vdots \\ \hat{E}(s(N)) \end{bmatrix}_{N\times 1}, \qquad \mathbf{\Theta} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_G \end{bmatrix}_{G\times 1} \qquad \mathbf{\Delta}_1 = \begin{bmatrix} \xi_1(s(1)) \\ \xi_1(s(2)) \\ \vdots \\ \xi_1(s(N)) \end{bmatrix}_{N\times 1},$$

where

$$\mathbf{\Psi} = \begin{bmatrix} s\hat{E}(s(1)) & s^2\hat{E}(s(1)) & \cdots & s^G \hat{E}(s(1)) & \\ s\hat{E}(s(2)) & s^2\hat{E}(s(2)) & \cdots & s^G \hat{E}(s(N)) & \\ \vdots & \vdots & & & \cdots \; \vdots \\ s\hat{E}(s(N)) & s_2\hat{E}(s(N)) & \cdots & s^G \hat{E}(s(N)) & \end{bmatrix}_{N\times G}.$$

Thus

$$\hat{F}(s) = \sum_{k=0}^{G} \hat{f}_k \, s^k = \hat{A}(s)^{-1} \, \hat{P}^{-1}(s)\hat{Q}(s),$$

(4) Form $\mathbf{Z}_F$ and $\mathbf{P}_F$ in equation (25) using the filtered data and apply the least-squares estimator to obtain the estimates $\hat{\mathbf{\Phi}}$.
(5) Go to step (2) and repeat until the estimates converge.

3.2. QUADRATIC NON-LINEAR CONTINUOUS-TERM RECONSTRUCTION

Next consider just the quadratic terms, setting $n = 2$ in equation (5) yields

$$H_2^{asym}(s_1, s_2) = \frac{1}{-\left[\sum_{l_1=0}^{L} c_{1,0}(l_1)(s_1 + \cdots + s_n)^{l_1}\right]} \left\{ \sum_{l_1 l_2=0}^{L} c_{0,2}(l_1, l_2)\,(s_1)^{l_1}(s_2)^{l_2} \right.$$

$$+ \sum_{l_1, l_2=0}^{L} c_{1,1}(l_1, l_2)\,(s_2)^{l_2}\, H_{1,1}(s_1)$$

$$\left. + \sum_{l_1, l_2=0}^{L} c_{2,0}(l_1, l_2)\; H_{2,2}^{asym}(s_1, s_2) \right\}. \tag{27}$$

With the recursive relation

$$H_{1,1}(s_1) = H_1(s_1)(s_1)^{l_1}$$

$$H_{2,2}^{asym}(s_1, s_2) = H_1(s_1)H_{1,1}(s_2)(s_1)^{l_2} = H_1(s_1)H_1(s_2)(s_2)^{l_1}(s_1)^{l_2},$$

where $H_1(\cdot)$ is the noise-free part in equation (18). In a practical implementation $H_1(\cdot)$ is formed using the coefficients $\hat{c}_{1,0}(\cdot)$ and $\hat{c}_{0,1}(\cdot)$ estimated in the linear-term identification.

Assuming that the kernel may be noisy and using the symmetrised formulation from equations (13) and (14) gives

$$
\begin{aligned}
&\hat{H}_2^{sym}(z_1, z_2)_{|z1 = e^{s1T}, z2 = e^{s2T}} \\
&= \hat{H}_2^{sym}(s_1, s_2) \\
&= H_2^{sym}(s_1, s_2) + N_2(e^{s1T} e^{s2T}) \\
&= \frac{1}{-[\sum_{l_1=0}^{L} c_{1,0}(l_1)(s_1+s_2)^{l_1}]} \left\{ \frac{1}{2} \sum_{l_1, l_2=0}^{L} c_{0,2}(l_1, l_2)[(s_1)^{l_1}(s_2)^{l_2} + (s_2)^{l_1}(s_1)^{l_2}] \right. \\
&\quad + \frac{1}{2} \sum_{l_1, l_2=0}^{L} c_{1,1}(l_1, l_2)[(s_2)^{l_2} H_{1,1}(s_1) + (s_1)^{l_2} H_{1,1}(s_2)] \\
&\quad \left. + \frac{1}{2} \sum_{l_1, l_2=0}^{L} c_{2,0}(l_1, l_2)[H_{2,2}^{asym}(s_1, s_2) + H_{2,2}^{asym}(s_2, s_1)]\right\} + N_2(s_1, s_2)] \\
&= \sum_{i=1}^{4L+7} \theta_i \, T(s_1, s_2) + N_2(s_1, s_2),
\end{aligned} \tag{28}
$$

where $\hat{H}_2^{sym}(z_1, z_2)$ is computed from the NARX model parameters as described in section 2 and

$$N_2(s_1, s_2) = \frac{W_2(s_1, s_2)}{Q_2(s_1, s_2)} \xi_2(s_1, s_2),$$

where $\xi_2(s_1, s_2)$ is a two-dimensional independent, zero mean white noise.

Notice that the coefficients $c_{1.0}(\cdot)$ have been estimated in the previous step where the linear terms were reconstructed, so equation (28) is linear-in-the-parameters. Unbiased least-squares estimates of the unknown coefficients can therefore be obtained by using a generalised least-squares-type algorithm as in the linear case. This consists of the following steps. Note that s_1 and s_2 are vectors consisting of data points over the s-plane. For simplicity, detailed expanded formulations are omitted here, but the algorithm consists of the following steps and is illustrated in the flow chart in Figure 2.

(1) Apply standard least squares to equation (28) to obtain estimates of $\hat{\theta}_i$, i.e., $\hat{c}_{0,2}(\cdot)$, $\hat{c}_{1,1}(\cdot)$ and $\hat{c}_{2,0}(\cdot)$, which will be biased if $N_2(s_1, s_2)$ is not white.
(2) Analyse the residual $\hat{N}_2(s_1, s_2)$ from equation (28):

$$\hat{N}_2(s_1, s_2) = \hat{H}_2^{sym}(s_1, s_2) - \sum_{i=1}^{4L+7} \hat{\theta}_i \, T(s_1, s_2). \tag{29}$$

(3) Estimate a filter $\hat{F}_2(s_1, s_2)$ as

$$\hat{F}_2(s_1, s_2)\hat{N}_2(s_1, s_2) = \xi_2(s_1, s_2) \tag{30}$$

using least squares so that

$$\hat{F}_2(s_1, s_2) = \hat{W}_2^{-1}(s_1, s_2)\hat{Q}_2(s_1, s_2).$$

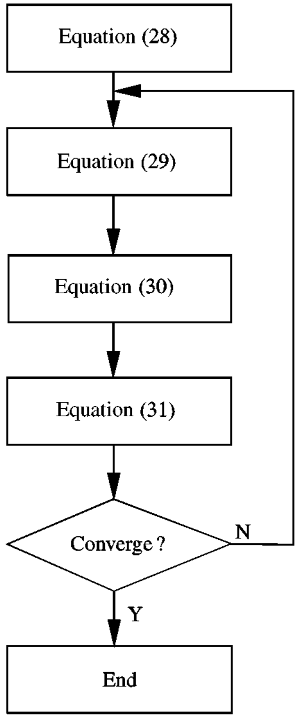

Figure 2. Flow chart of the reconstruction procedure for continuous-time quadratic non-linear term.

(4) Multiply both sides of equation (28) by $\hat{F}_2(s_1, s_2)$,

$$\hat{F}_2(s_1, s_2)\, \hat{H}_2^{sym}(s_1, s_2) = \sum_{i=1}^{4L+7} \theta_i\, \hat{F}_2(s_1, s_2)\, T(s_1, s_2) + \xi_2(s_1, s_2) \tag{31}$$

and apply least squares to get estimates of the parameters $\hat{\theta}_i$, i.e., $\hat{c}_{0,2}(\cdot)$, $\hat{c}_{1,1}(\cdot)$, $\hat{c}_{2,0}(\cdot)$.

(5) Go to step (2) and repeat until the estimates converge.

This procedure can be continued for higher order non-linearities, $n = 3,4, \ldots$, etc.

3.3. MODEL STRUCTURE DETERMINATION

The sequential construction of the model starting with the linear terms, followed by the quadratic terms, and so on as described in the previous subsections forms the basis of the solution. But in practice only a few of the numerous possible candidate linear, quadratic, cubic, etc., terms will be relevant. It is therefore important, when no *a priori* information is available regarding the continuous-time model, to be able to select significant model terms at each stage of the model reconstruction. This can be achieved using a modification of the orthogonal least-squares method (OLS) [17].

Consider a system expressed by the linear-in-the-parameters model

$$z = \sum_{i=1}^{M} \theta_i p_i + \varepsilon, \tag{32}$$

where θ_i, $i = 1, \ldots, M$ are unknown parameters.

Reformulating equation (32) in the form of an auxiliary model yields

$$z = \sum_{i=1}^{M} g_i w_i + \varepsilon, \tag{33}$$

where g_i, $i = 1, \ldots, M$ are auxiliary parameters and w_i, $i = 1, \ldots, M$ are constructed to be orthogonal over the data record such that

$$\sum_{t=1}^{N} w_j(t) w_{k+1}(t) = 0, \quad j = 0, 1, \ldots, k, \tag{34}$$

where N is the length of the data record.

Multiplying the auxiliary model (33) by itself, using the orthogonal property (34) and taking the time average gives

$$\frac{1}{N} \sum_{t=1}^{N} z^2(t) = \frac{1}{N} \sum_{t=1}^{N} \left\{ \sum_{j=0}^{M} g_i^2 \, w_i^2(t) \right\} + \frac{1}{N} \sum_{t=1}^{N} \varepsilon^2(t). \tag{35}$$

Finally, define

$$ERR_i = \frac{\sum_{l=1}^{N} g_i^2 w_i^2(t)}{\sum_{t=1}^{N} z^2(t) - (1/N)\{\sum_{t=1}^{N} z(t)\}^2} \times 100 \tag{36}$$

for $i = 1, 2, \ldots, M$. The quantity ERR_i is called the error reduction ratio and provides an indication of which terms should be included in the model in accordance with the contribution each term makes to the energy of the dependent variable. Terms with associated ERR values which are less than a pre-defined threshold value (e.g., 0·01) can be considered to be insignificant and negligible.

However, this idea cannot be applied directly to the iterative identification procedures described in section 3. In general, the result of the first iteration will be biased and this will not give the correct significance of each term. Some modification must therefore be made when implementing OLS in this particular application. Simulations suggest that the best solution to this problem is to begin with an overparameterized model structure. When the parameters of this model converge, the terms where the ERR values are below the threshold are then eliminated. Finally, re-estimate the parameters for this reduced model structure and hence obtain the final coefficients. This ideas is illustrated in the following simulation example.

The selection of the order of the filters $\hat{F}(\cdot)$ and $\hat{F}_2(\cdot, \cdot)$ is also important, and the OLS algorithm can also be used to determine these orders.

4. SIMULATION EXAMPLE

Consider the non-linear system

$$1\, y + 0{\cdot}002\, Dy + 0{\cdot}0001\, D^2 y - 1\, u + 0{\cdot}1\, y^2 - 0{\cdot}006\, yDy = 0. \tag{37}$$

This model was simulated using MATLAB. The input signal was chosen to be a random sequence with amplitude ± 1, and 1000 input-output data were collected after sampling at 400 HZ. A white noise was then added to the output to give a SNR of 20 dB.

4.1. NARMAX IDENTIFICATION

The first step in the identification procedure is to identify a NARMAX model of the system. An enlarged model structure was used to represent the system, and after passing all the model validation tests [18] the final model was obtained as

$$\begin{aligned}
y(k) = {} & 0{\cdot}22099y(k-1) - 0{\cdot}28266y(k-8) + 0{\cdot}00073y(k-2) + 0{\cdot}11415u(k-2) \\
& + 0.13593u(k-3) + 0{\cdot}16375u(k-5) + 0{\cdot}09909y(k-3) + 0{\cdot}13737u(k-4) \\
& + 0{\cdot}02375y(k-3)y(k-8) + 0{\cdot}08582y(k-9) + 0.13896u(k-7) + 0{\cdot}04719u(k-1) \\
& - 0{\cdot}34567y(k-12) + 0{\cdot}11822u(k-6) - 0{\cdot}0840y(k-7)y(t-9) + 0{\cdot}13378y(k-1)y(k-2) \\
& + 0{\cdot}09797u(k-8) + 0{\cdot}08407u(k-9) - 0{\cdot}06159y(k-13) + 0{\cdot}07489u(k-10) \\
& - 0{\cdot}09918y(k-10)y(k-11) + 0{\cdot}03417u(k-11) - 0{\cdot}09156y(k-11) + 0{\cdot}1092y(k-5) \\
& + 0{\cdot}08644y(k-10) + 0{\cdot}10144y(k-7)y(k-15) - 0{\cdot}10199y(k-9)y(k-14) \\
& + 0{\cdot}01344u(k-12) - 0{\cdot}02788y(k-16) - 0{\cdot}09730y(k-2)y(k-8) + \Theta_\xi + \xi(k),
\end{aligned} \tag{38}$$

where Θ_ξ represents the noise model terms. Discarding the noise model terms Θ_ξ which were included to ensure unbiased process model parameters and $\xi(k)$ in equation (38), $H_1(z)$ and the asymmetric form of $H_2(z_1, z_2)$ can be computed directly from the parameters of the NARX model as

$$H_1(z) = \frac{\begin{aligned}&[0{\cdot}04719z^{-1} + 0{\cdot}11415z^{-2} + 0{\cdot}13593z^{-3} + 0{\cdot}13737z^{-4} + 0{\cdot}16375z^{-5} + 0{\cdot}11822z^{-6} \\ &+ 0{\cdot}13896z^{-7} + 0{\cdot}097974z^{-8} + 0{\cdot}084072z^{-9} + 0{\cdot}074886z^{-10} + 0{\cdot}03417z^{-11} + 0{\cdot}01344z^{-12}]\end{aligned}}{\begin{aligned}&[1 - 0{\cdot}22099z^{-1} - 0{\cdot}00073z^{-2} - 0{\cdot}09909z^{-3} - 0.10916z^{-5} + 0{\cdot}28266z^{-8} \\ &+ 0{\cdot}08582z^{-9} - 0{\cdot}08644z^{-10} + 0{\cdot}09156z^{-11} + 0{\cdot}34567z^{-12} + 0{\cdot}06159z^{-13} + 0{\cdot}02788z^{-16}]\end{aligned}} \tag{39}$$

and

$$H_2^{asym}(z_1, z_2) = H_1(z_1)H_1(z_2) \times \frac{\begin{aligned}&[0{\cdot}02375z_1^{-3}z_2^{-8} - 0{\cdot}0840z_1^{-7}z_2^{-9} + 0{\cdot}13378z_1^{-1}z_2^{-2} - 0{\cdot}09918z_1^{-10}z_2^{-11} + 0{\cdot}10144z_1^{-7}z_2^{-15} \\ &- 0{\cdot}10199z_1^{-9}z_2^{-14} - 0{\cdot}09730z_1^{-2}z_2^{-8}]\end{aligned}}{\begin{aligned}&[1 - 0{\cdot}22099(z_1+z_2)^{-1} - 0{\cdot}00073(z_1+z_2)^{-2} - 0{\cdot}09909(z_1+z_2)^{-3} - 0{\cdot}10916(z_1+z_2)^{-5} \\ &+ 0{\cdot}28266(z_1+z_2)^{-8} + 0{\cdot}08582(z_1+z_2)^{-9} - 0{\cdot}08644(z_1+z_2)^{-10} + 0{\cdot}09156(z_1+z_2)^{-11} \\ &+ 0{\cdot}34567(z_1+z_2)^{-12} + 0{\cdot}06159(z_1+z_2)^{-13} + 0{\cdot}02788(z_1+z_2)^{-16}].\end{aligned}} \tag{40}$$

The non-linear differential equation model can now be constructed sequentially. Just the linear model terms are identified first, followed by the quadratic non-linear terms and so on. At each step the algorithm determines the appropriate model terms and produces estimates of the unknown parameters.

4.2. LINEAR TERM RECONSTRUCTION

A total of 500 $\hat{H}_1(z)$ data points with $z = e^{sT}$ were generated in equation (39) choosing $s = [t_1, jt_2]$ where t_1, t_2 were selected as random points over 0–500.

An initial overparameterized structure was used with five linear input terms and three linear output terms. The results using the iteration procedure in section 3.1 are listed in Table 2. The ERR values obviously suggest that the extra terms $D^4 y$, $D^3 y$, Du and $D^2 u$ can be removed from the model structure. Eliminating these terms and re-applying the estimator provide the final results in Table 3. A comparison of the results in Tables 2 and 3 shows that the estimates from the first iteration were biased as expected.

The order of the filter $\hat{F}(s)$ was determined based on the ERR values obtained when estimating the filter $\hat{F}(s)\hat{E}_1(s) = \xi_1(s)$. When the order was set to be 10, the sum of the ERR values was 99·992% suggesting that the order of the filter was adequate and $\hat{F}(s)\hat{E}_1(s)$ should be white. Figure 3 shows a comparison of the autocorrelation of the true $N_1(s)$ and the estimated $\hat{N}_1(s)$. Figure 4 shows the autocorrelation of the estimated $\hat{E}_1(s)$ and the estimated $\hat{\xi}_1(s) = \hat{F}(s)\hat{E}_1(s)$. It can be seen that $\hat{E}_1(s)$ has been reduced to white $\hat{\xi}_1(s)$ by the operation of the filter $\hat{F}(s)$.

4.3. NON-LINEAR TERM RECONSTRUCTION

The data points were generated from equation (40) with $z_1 = e^{s_1 T}$, $z_2 = e^{s_2 T}$ along s_1, $s_2 = [t_{11}, jt_{22}]$ where t_{11}, t_{22} are random points between 0 and 220. A total of 70 points were chosen along both axes. Initially, an overparameterised non-linear model with model

TABLE 2

Initial identification resutls based on an overparameterised model structure for the linear-term reconstruction

Terms	$D^4 y$	$D^3 y$	$D^2 y$	Dy	u	Du	$D^2 u$
Estim's	$3{\cdot}12 \times 10^{-10}$	$2{\cdot}885 \times 10^{-9}$	$1{\cdot}03 \times 10^{-4}$	$1{\cdot}891 \times 10^{-3}$	$-0{\cdot}961$	$-7{\cdot}23\times10^{-5}$	$-1{\cdot}83\times10^{-6}$
ERR (%)	$2{\cdot}99 \times 10^{-3}$	$2{\cdot}23 \times 10^{-7}$	26·99	1·107	71·90	$4{\cdot}70\times10^{-5}$	$5{\cdot}63\times10^{-4}$

TABLE 3

Final linear-term identification

Terms	$D^2 y$	Dy	u
Estim's of 1st itera	$7{\cdot}5832 \times 10^{-05}$	$1{\cdot}8424 \times 10^{-3}$	− 0·84392
Estim's converged	0·00009951	0·0019168	− 0·96325
True value	0·0001	0·002	− 1·00
ERR (%)	63·8562	1·7937	34·3461

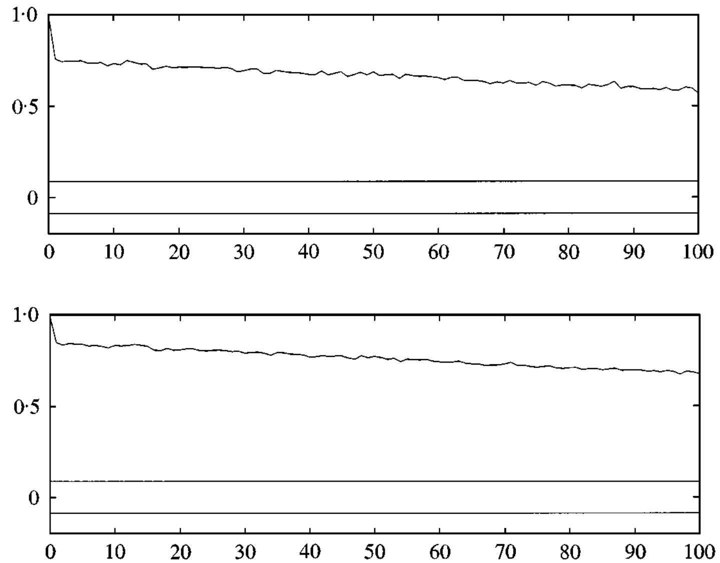

Figure 3. Autocorrelation text of $N_1(s)$ (upper) and $\hat{N}_1(s)$ (lower).

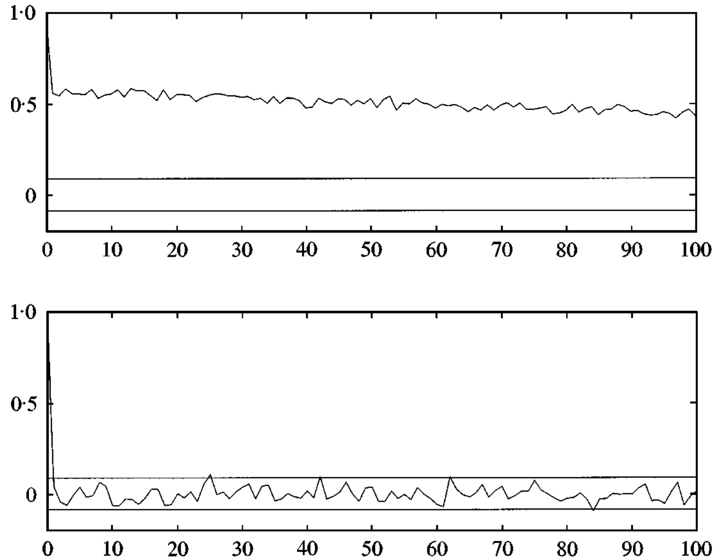

Figure 4. Autocorrelation test of $\hat{E}_1(s)$ (upper) and $\hat{\xi}_1(s)$ (lower).

terms y^2, yDy, $DyDy$ and yu was used. Applying the iterative procedure in section 3.2 and retaining the two most significant terms produced the results illustrated in Table 4. The sum of ERR values of 99·655% implies that the terms y^2 and yDy are sufficient to represent the non-linear phenomena.

5. IDENTIFICATION OF AN ELECTROMAGNETIC BEARING SYSTEM

The data used in this example was collected from a flywheel energy storage unit for an electric car. The main component in this unit is the electromagnetic bearing system

TABLE 4

Final quadratic non-liner term identification

Terms	Estim's of 1st itera	Estim's converged	True value	EER(%)
y^2	0·0794	0·0990	0·1	17·184
yDy	− 0·0056	− 0·0055	− 0·006	82·471
SUM (ERR)%				99·655

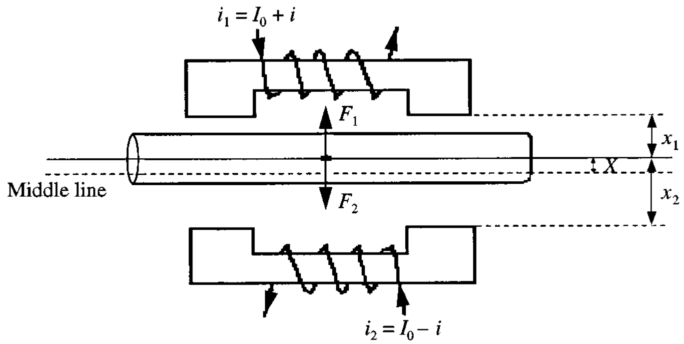

Figure 5. Schematic diagram of the electromagnet system.

illustrated in Figure 5. It is known that a quadratic non-linearity usually relates the force F_n and the input currents i_n, $n = 1{,}2$ in Figure 5. A continuous-time model is required for this system so that the designers can relate the system components to the model and to produce more insight for subsequent controller design studies. The block diagram of the experimental set-up is illustrated in Figure 6, where $r(t)$ is a random signal that was added for the purpose of identification. The output $x(t)$ and the input $i(t)$ were measured, as pointed in Figure 6 and shown in Figures 7 and 8, at the sampling time interval $1{\cdot}5 \times 10^{-4}$ s.

The input–output data was decimated to give an effective sampling time interval of $4{\cdot}5 \times 10^{-4}$ s. A quadratic NARMAX model with only output non-linear terms was identified.

The reconstructed continuous-time non-linear model is derived in equation (41). A comparison of the linear part of the reconstructed continuous-time model and the NARX model is illustrated in Figure 9 and this shows that the mapping on the imaginary and the real axis are recovered with very little error. Figure 10 shows the comparison of the quadratic non-linear frequency response of the reconstructed continuous-time model and the NARX model. Finally, a comparison of the measured output and the simulated output from reconstructed continuous-time model with the same input signal at the original sampling interval of $1{\cdot}5 \times 10^{-4}$ s is illustrated in Figure 8. This comparison is only possible because the continuous-time model can be simulated for any sample interval.

$$D^{12}x + 1{\cdot}404695\times10^{3} D^{11}x + 1{\cdot}595366\times10^{7} D^{10}x + 1{\cdot}978458\times10^{10} D^{9}x + 9{\cdot}027335\times10^{13} D^{8}x$$
$$+ 9{\cdot}283008\times10^{16} D^{7}x + 2{\cdot}199978\times10^{20} D^{6}x + 1{\cdot}654813\times10^{23} D^{5}x + 2{\cdot}255742\times10^{26} D^{4}x$$
$$+ 9{\cdot}989155\times10^{28} D^{3}x + 7{\cdot}261332\times10^{31} D^{2}x + 9{\cdot}738196\times10^{33} Dx + 2{\cdot}418980\times10^{36} x$$

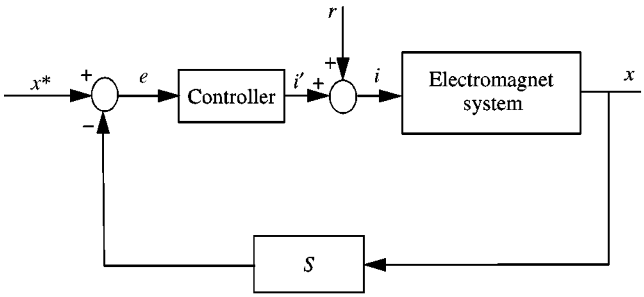

Figure 6. Block diagram of the controlled system.

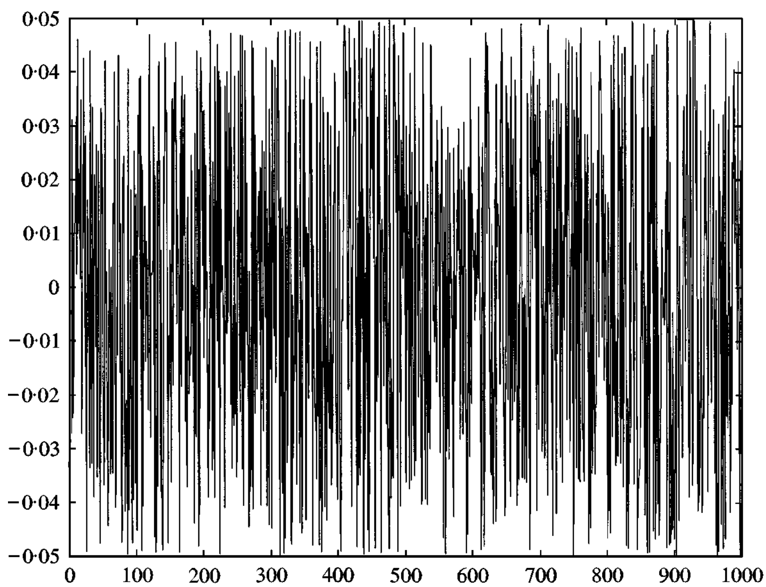

Figure 7. Input signal $i(t)$.

$$
\begin{aligned}
&- 26{\cdot}02691 D^{11} i + 2{\cdot}147129 \times 10^{04} D^{10} i - 4{\cdot}316928 \times 10^{8} D^{9} i + 1{\cdot}120871 \times 10^{11} D^{8} i \\
&- 2{\cdot}637928 \times 10^{15} D^{7} i - 8{\cdot}1704940 \times 10^{17} D^{6} i - 7{\cdot}184292 \times 10^{21} D^{5} i - 5{\cdot}484130 \times 10^{24} D^{4} i \\
&- 8{\cdot}640049 \times 10^{27} D^{3} i - 8{\cdot}046999 \times 10^{30} D^{2} i - 3{\cdot}588718 \times 10^{33} D i - 3{\cdot}025046 \times 10^{36} i \\
&- 2{\cdot}303532 \times 10^{36} i^{2} - 1{\cdot}373715 \times 10^{31} i D^{2} i + 2{\cdot}927454 \times 10^{22} D i D^{4} i - 8{\cdot}099636 \times 10^{17} D i D^{5} i \\
&+ 1{\cdot}646983 \times 10^{33} i D i + 1{\cdot}880561 \times 10^{23} D^{2} i D^{3} i - 8{\cdot}363859 \times 10^{24} i D^{4} i \\
&+ 4{\cdot}383683 \times 10^{19} D^{2} i D^{4} i - 0{\cdot}029673 D^{6} i D^{7} i - 1{\cdot}664776 \times 10^{5} D^{4} i D^{7} i = 0.
\end{aligned}
$$

6. CONCLUSIONS

A new algorithm for reconstructing linear and non-linear differential equation models from sampled data by identifying a non-linear difference model has been proposed as

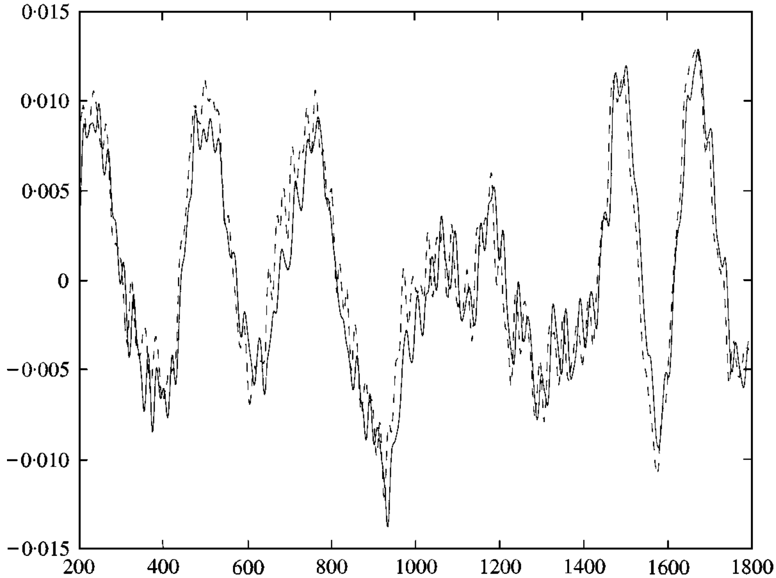

Figure 8. A comparison of the measured output and the simulated output from the reconstructed continuous-time model for the electromagnetic suspension system at the original sampling interval

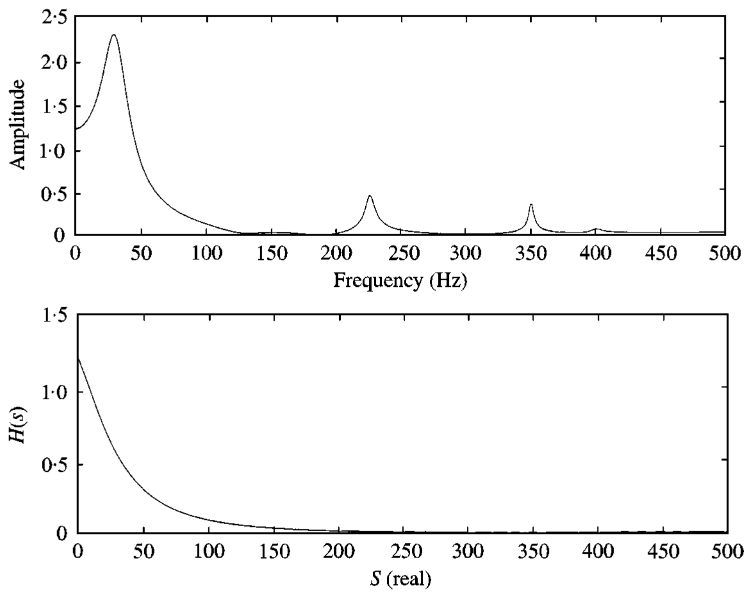

Figure 9. Comparison of linear part of the reconstructed continuous-time model and the NARX model for the electromagnetic suspension system: upper-comparison along the imaginary axis (solid—NARX, dashed—continuous); lower-comparison along the real axis (solid—NARX, dashed-continuous).

a practical means of implementing the kernel invariance procedure. It has been shown that by combining the procedures of generalised least squares, with the orthogonal estimator and the error reduction ratio that the parameters and the structure of non-linear differential equation models can be identified without the need to compute higher order derivatives of noisy data.

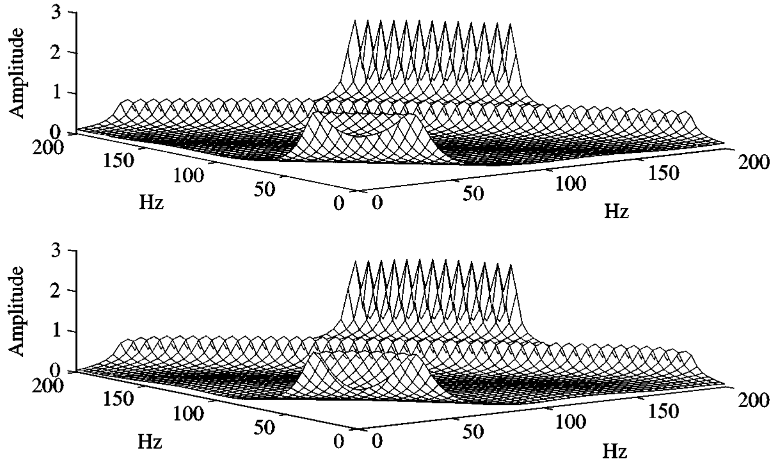

Figure 10. Comparison of quadratic frequency response of the reconstructed continuous-time model and the NARX model for the electromagnetic suspension system: upper—from NARX: lower-from the identified continuous time model.

ACKNOWLEDGMENTS

SAB gratefully acknowledges that part of this work was sponsored by EPSRC. LL gratefully acknowledges financial support from an ORS award and a scholarship from the University of Sheffield. We would also like to thank Dr Z. Wu of the Department of EEE, University of Sheffield for the data associated with the electromagnetic bearing system.

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APPENDIX A: NOMENCLATURE

ARMAX	auto-regressive moving average model with exogenous input
ERR	error reduction ratio
KIA	kernal invariance algorithm
KIM	kernel invariance method
NARMAX	non-linear auto-regressive moving average model with exogenous input
NARX	non-linear auto-regressive model with exogenous input
NDE	non-linear differential equation
OLS	orthogonal least squares